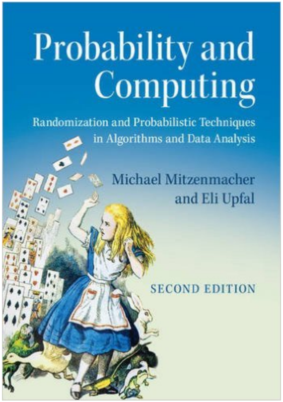
CS155/254: Probabilistic Methods in Computer Science

Chapter 15: Pairwise Independent and Hashing



Pairwise Independence

Definition

1. A set of events $E_1, E_2, \ldots E_n$ is k-wise independent if for any subset $I \subseteq [1, n]$ with $|I| \leq k$,

$$\Pr\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} \Pr(E_i).$$

2. A set of random variables $X_1, X_2, \ldots X_n$ is *k-wise independent* if for any subset $I \subseteq [1, n]$ with $|I| \leq k$, and any values x_i, $i \in I$,

$$\Pr\left(\bigcap_{i \in I} X_i = x_i\right) = \prod_{i \in I} \Pr(X_i = x_i).$$

If true for $k = n$ the random variables are *mutually independent*.

Pairwise Independent

Definition

The random variables $X_1, X_2, \ldots X_n$ are said to be *pairwise independent* if they are 2-wise independent. That is, for any pair i, j and any values a, b,

$$\Pr((X_i = a) \cap (X_j = b)) = \Pr(X_i = a) \cdot \Pr(X_j = b).$$

Application: We can construct $m = 2^b - 1$ uniform pairwise independent 0-1 random variable from b independent, uniform random bits, $X_1, \ldots, X_b$.
$m = 2^b - 1$ uniform pairwise independent 0-1 random variable in a sample space with 2^b simple events.

Construction of Pairwise Independent Bits

We are given b independent, uniform random bits, $X_1, \ldots, X_b$.

Let $S_1, \ldots, S_{2^b-1}$ be an arbitrary order of all the non-empty subsets of $\{1, 2, \ldots, b\}$.

Let $\oplus$ be the exclusive-or operation. Define $m = 2^b - 1$ random variables

$$Y_j = \oplus_{i \in S_j} X_i = \sum_{i \in S_j} X_i \bmod 2$$

- $Pr(Y_i = 1) = Pr(Y_i = 0) = 1/2$. Let $z \in S_i$. Fix the bits in $S_i - \{z\}$. The value of Y_i is determined by the value of z.
- Pairwise independence: For any $c, d \in \{0, 1\}$

  $$\Pr((Y_k = c) \cap (Y_\ell = d)) = \Pr(Y_\ell = d \mid Y_k = c) \cdot \Pr(Y_k = c) = 1/4.$$

  Since the value of Y_ℓ is determined by $z \in S_\ell \setminus S_k$

Thus, $Y_1, \ldots Y_{2^b-1}$ are pairwise independent, uniform $\{0, 1\}$ random variables.

The Expectation Argument: Large Cut-Set in a Graph.

Theorem

Given any graph $G = (V, E)$ with n vertices and m edges, there is a partition of V into two disjoint sets A and B such that at least $m/2$ edges connect a vertex in A to a vertex in B.

Let $Y_1 \ldots, Y_n$ pairwise independent uniform $\{0, 1\}$ random variables, generated from $\log_2 n + 1$ independent random bits.

Place such that vertex i is in set A if $Y_i = 0$ else vertex i is placed in set B.

Let $Z_e = 1$ if edge e crosses the cut, and $Z_e = 0$ otherwise.

Let $e = \{i, j\}$, then $\Pr(Z_e = 1) = \Pr(Y_i \neq Y_j) = \frac{1}{2}$,

$\mathbf{E}[Z] = \mathbf{E}\left[\sum_{i=1}^{m} Z_i\right] = \sum_{i=1}^{m} \mathbf{E}[Z_i]$, the sample space has an assignment with a cut $\geq m/2$.

The sample space has only $2n$ simple event, algorithm can try all simple events to find a good assignment.

Independent Set in a Graph

An *independent set* in a graph G is a set of vertices with no edges between them.

Theorem

Let $G = (V, E)$ be a graph on n vertices with $dn/2$ edges. Then G has an independent set with at least $n/2d$ vertices.

Algorithm:

1. Delete each vertex of G (together with its incident edges) independently with probability $1 - 1/d$.
2. For each remaining edge, remove it and one of its adjacent vertices.

$X =$ number of vertices that survive the first step of the algorithm.

$$E[X] = \frac{n}{d}.$$

$Y =$ number of edges that survive the first step.
An edge survives if and only if its two adjacent vertices survive.

$$E[Y] = \frac{nd}{2}\left(\frac{1}{d}\right)^2 = \frac{n}{2d}.$$

The second step of the algorithm removes all the remaining edges, and at most Y vertices.
Size of output independent set:

$$E[X - Y] = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d}.$$

The n events of deleting nodes need only to be pairwise independent events.

Deterministic Algorithm for Independent Set

Theorem

Let $G = (V, E)$ be a graph on n vertices with $dn/2$ edges. We can compute in $O(n^{\log_2 d})$ steps an independent set in G with at least $n/2d$ vertices.

The existence proof required n pairwise 0-1 independent events with probabilities $1/d, 1 - 1/d$.

If $n + 1$ and $d + 1$ are powers of two, we can use $\log_2 d$ independent sets of n pairwise independent bits, The $\log_2(n + 1)$ random bits used for generating each of the $\log_2 d$ sets are independent.

For $i = 1, \ldots, \log_2 n$ and $j = 1, \ldots n$, let Y_j^i be the random bit of vertex j in system i.

Delete vertex j if all its $\log_2 d$ bits are 1. Probability that a node is deleted is $1/d$.

For each i and $j \neq k$, $Pr(Y_j^i = 1 \cap Y_k^i = 1) = 1/4$. Since the $\log_2 d$ sets are independent, the probability that vertices j and k are deleted is $1/d^2$.

The sample space was generated with a total of $\log_2 d \log_2(n + 1)$ bits. It has $(n + 1)^{\log_2 d}$ events. We can check all of them to find an independent set with the required size.

Deviation Bound

You cannot use Chernoff bound but you can use Chebyshev bound.

Theorem

Let $X = \sum_{i=1}^{n} X_i$, where the X_i are pairwise independent random variables. Then

$$\mathbf{Var}[X] = \sum_{i=1}^{n} \mathbf{Var}[X_i].$$

Proof: $\mathbf{Var}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbf{Var}[X_i] + 2\sum_{i<j} \mathbf{Cov}(X_i, X_j)$.

For Pairwise independent $X_i, X_2, \ldots, X_n$,

$$\mathbf{Cov}(X_i, X_j) = \mathbf{E}[(X_i - \mathbf{E}[X_i])(X_j - \mathbf{E}[X_j])] = \mathbf{E}[X_i X_j] - \mathbf{E}[X_i]E[X_j] = 0.$$

Corollary

Let $X = \sum_{i=1}^{n} X_i$, where the X_i are pairwise independent random variables. Then

$$\Pr(|X - \mathbf{E}[X]| \geq a) \leq \frac{\mathbf{Var}[X]}{a^2} = \frac{\sum_{i=1}^{n} \mathbf{Var}[X_i]}{a^2}.$$

Perfect Hashing

We want to store n records using minimus space and minimum retrieval (search) time.

We can store the n records in a sorted order. Space $= O(n)$, retrieval time $= O(\log n)$

We can hash the n keys to a table of size $O(n)$, with $O(1)$ expected retrieval time, and $O(\log n)$ expected maximum retrieval time. (We need a table of size $\Omega(n^{1+\epsilon})$ for expected maximum $1/\epsilon$.)

Goal: Store a **static dictionary** of n items in a table of $O(n)$ space such that any search takes $O(1)$ time.

Static dictionary - any insert or delete operation requires rearranging the entire table.

Universal hash functions

Definition

Let U be a universe with $|U| \geq n$ and $V = \{0, 1, \ldots, n-1\}$. A family of hash functions $\mathcal{H}$ from U to V is said to be k-*universal* if, for any elements $x_1, x_2, \ldots, x_k$, when a hash function h is chosen uniformly at random from $\mathcal{H}$,

$$\Pr(h(x_1) = h(x_2) = \ldots = h(x_k)) \leq \frac{1}{n^{k-1}}.$$

If $\Pr(h(x_1) = h(x_2) = \ldots = h(x_k)) = \frac{1}{n^{k-1}}$, then for any $x_1, x_2, \ldots, x_k$ the random variables $h(x_1), \ldots, h(x_k)$ are k-pairwise independent.

Example of 2-Universal Hash Functions

Universe $U = \{0, 1, 2, \ldots, m-1\}$
Table keys $V = \{0, 1, 2, \ldots, n-1\}$, with $m \geq n$.
A family of hash functions obtained by choosing a prime $p \geq m$,

$$h_{a,b}(x) = ((ax + b) \bmod p) \bmod n,$$

and taking the family

$$\mathcal{H} = \{h_{a,b} \mid 1 \leq a \leq p-1, 0 \leq b \leq p\}.$$

Lemma

$\mathcal{H}$ *is 2-universal.*

Lemma

$\mathcal{H}$ is 2-universal.

Proof: We first observe that for $x_1, x_2 \in \{0, \ldots, p-1\}$, $x_1 \neq x_2$,

$$ax_1 + b \neq ax_2 + b \mod p.$$

Thus, if $h_{a,b}(x_1) = h_{a,b}(x_2)$ there is a pair (s, r) such that,

1. $(ax_1 + b) \mod p = r$
2. $(ax_2 + b) \mod p = s$
3. $s \neq r$, $s = (r \mod n)$

For each r there are $\leq \lceil \frac{p}{n} \rceil - 1$ values $s \neq r$ such that $s = (r \mod n)$, and for each pair (r, s) there is only one pair (a, b) that satisfies the relation.

Over all the $p(p-1)$ choice of (a, b), r gets p different values.

Thus, the probability of a collision is $\leq \frac{p(\lceil \frac{p}{n} \rceil - 1)}{p(p-1)} \leq \frac{1}{n}$.

Lemma

If $h \in \mathcal{H}$ is chosen uniformly at random from a 2-universal family of hash functions mapping the universe U to $[0, n-1]$, then for any set $S \subset U$ of size m, with probability $\geq 1/2$ the number of collisions is bounded by m^2/n.

proof:

Let $s_1, s_2, \ldots, s_m$ be the m items of S. Let X_{ij} be 1 if the $h(s_i) = h(s_j)$ and 0 otherwise. Let $X = \sum_{1 \leq i < j \leq n} X_{ij}$.

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{1 \leq i < j \leq n} X_{ij}\right] = \sum_{1 \leq i < j \leq m} \mathbf{E}[X_{ij}] \leq \binom{m}{2}\frac{1}{n} < \frac{m^2}{2n},$$

Markov's inequality yields

$$\Pr(X \geq m^2/n) \leq \Pr(X \geq 2\mathbf{E}[X]) \leq \frac{1}{2}.$$

Definition

A hash function is perfect for a set S if it maps S with no collisions.

Lemma

If $h \in \mathcal{H}$ is chosen uniformly at random from a 2-universal family of hash functions mapping the universe U to $[0, n-1]$, then for any set $S \subset U$ of size m, such that $m^2 \leq n$ with probability $\geq 1/2$ the hash function is perfect

$$\Pr(X \geq 1) \leq \Pr(X \geq m^2/n) \leq \Pr(X \geq 2\mathbf{E}[X]) \leq \frac{1}{2}.$$

Theorem

The two-level approach gives a perfect hashing scheme for m items using $O(m)$ bins.

Level I: use a hash table with $n = m$. Let X be the number of collisions,

$$\Pr(X \geq m^2/n) \leq \Pr(X \geq 2\mathbf{E}[X]) \leq \frac{1}{2}.$$

When $n = m$, there exists a choice of hash function from the 2-universal family that gives at most m collisions.

Level II: Let c_i be the number of items in the i-th bin. There are $\binom{c_i}{2}$ collisions between items in the i-th bin, thus

$$\sum_{i=1}^{m} \binom{c_i}{2} \leq m.$$

For each bin with $c_i > 1$ items, we find a second hash function that gives no collisions using space c_i^2. The total number of bins used is bounded above by

$$m + \sum_{i=1}^{m} c_i^2 \leq m + 2\sum_{i=1}^{m} \binom{c_i}{2} + \sum_{i=1}^{m} c_i \leq m + 2m + m = 4m.$$

Hence the total number of bins used is only $O(m)$.

Perfect Hashing

Theorem

There is a storage method that can store m keys in a table of size $O(m)$ with $O(1)$ search time.